NP = PSPACE

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Abstract. **NP = PSPACE** is provable by proof-compression techniques.

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1 Introduction

We assume that the reader is familiar with basics of computational complexity theory where the converse of **NP** $\subseteq$ **PSPACE** is one of the open problems. The attached affirmative proof (Chapters 2–4) is due to the first author, but it was the second author who posed an initial idea to attack a weaker conjecture **NP = coNP** by reductions in Prawitz-style natural deduction formalisms for propositional logic. That idea included interactive use of minimal, intuitionistic and classical formalisms, so its practical implementation turned out to be rather involved. On the contrary, the attached proof runs in the minimal logic via embedding Hudelmaier's cutfree sequent calculus into Prawitz's calculus of natural deductions with dag-like compressions playing a crucial role.

In our earlier proof-compression research [2], [3], [4] dealing with sequent calculi [1] we obtained such basic result (et al):

Any tree-like deduction ∂ of any given sequent S is constructively compressible to a dag-like deduction ∂_{C} of S in which sequents occur at most once. I.e., in ∂_{C}, distinct nodes are supplied with distinct sequents (that also occur in ∂).

Loosely speaking it shows that sequent calculi (whether minimal, intuitionistic or classical) admit full dag-like compression. However, even in the case of cutfree sequent calculi having good proof search and other nice properties (like Gentzen's subformula property), this result still gives us no polynomial control over the size of the compressed dag-like deductions. The reason is that sequents can be viewed as sets of (sub)formulas, which makes their number grow exponential in the size of the conclusion. Now natural deductions consist of single formulas, so there was hope to overcome this problem. On the other hand, full

[1] Also note that [8] shows a mimp-like formalization of natural deductions that admits "explicit" and size-preserving strong normalization procedure.

dag-like compression that identifies arbitrary nodes supplied with equal formulas is problematic, as there is a risk of confusion between deduced formulas and the same formulas used above as discharged assumptions. But what can be done is a sort of horizontal compressing. The underlying idea is that if a given tree-like deduction has merely polynomial height and every horizontal section is dag-like compressible to the nodes supplied with polynomially many pairwise distinct formulas, then the horizontal dag-like compression should have polynomial size. In the sequel we elaborate this program. [2]

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2 Overview of the proof

1. Formalize purely implicational minimal propositional logic [7] as fragment (call it $\mathrm{LM}_{\rightarrow}$) of Hudelmaier's [6] tree-like cutfree sequent calculus for the intuitionistic one. Note that for any $\mathrm{LM}_{\rightarrow}$ proof ∂ of a given formula α:
   (a) the height of ∂ (abbr.: $h(\partial)$) is polynomial (actually linear) in the length of α (abbr.: $|\alpha|$),
   (b) the total number of pairwise distinct formulas occurring in ∂ (abbr.: $\phi(\partial)$) is also polynomial (actually quadratic) in $|\alpha|$.
2. Embed $\mathrm{LM}_{\rightarrow}$ into Prawitz's [9] tree-like natural deduction formalism for minimal logic (call it $\mathrm{NM}_{\rightarrow}$). Observe that this translation preserves polynomial estimates (a) and (b).
3. Elaborate the notion of dag-like deducibility in $\mathrm{NM}_{\rightarrow}$.
4. Elaborate and apply *horizontal tree-to-dag proof compression* in $\mathrm{NM}_{\rightarrow}$. Note that for any given tree-like input ∂, the size of such dag-like compression is bounded by the product of $h(\partial)$ and $\phi(\partial)$. Hence, in the dag-like version of $\mathrm{NM}_{\rightarrow}$, the size of the compressed embedded tree-like $\mathrm{LM}_{\rightarrow}$ proof of α is polynomially bounded in $|\alpha|$. Since purely implicational minimal propositional logic is known to be PSPACE-complete [11], [12], conclude: **NP = PSPACE** (and beyond).

[2] The second author's initial idea (see above) was to transform a given super-polynomially sized tree-like deduction in purely implicational fragment of Prawitz's calculus for minimal logic into a suitable tree-like deduction of polynomial height in extended calculus for classical logic and proceed further by proof compression techniques (see also Appendix B). Together with familiar negative translations into implicational fragments of the intuitionistic and/or minimal logic this should have inferred **NP = coNP**. That program was superseeded by Hudelmaier's bounds for the intuitionistic and/or minimal logic in order to strengthen the conclusion to **NP = PSPACE**.

3 More detailed exposition

3.1 Sequent calculus $\mathrm{LM}_{\rightarrow}$.

Definition 1 $\mathrm{LM}_{\rightarrow}$ *includes the following axioms* $(\mathrm{M}A)$ *and inference rules* $(\mathrm{M}I1\rightarrow)$, $(\mathrm{M}I2\rightarrow)$, $(\mathrm{M}E\rightarrow P)$, $(\mathrm{M}E\rightarrow\rightarrow)$ *in standard intuitionistic sequent formalism* [3] *of one propositional connective* $\rightarrow$ *(*α,β,γ *formulas;* p*,* q *distinct propositional variables; in* $(\mathrm{M}I1\rightarrow)$*, no* $(\alpha\rightarrow\beta)\rightarrow\gamma$ *occurs in* Γ*, while in* $(\mathrm{M}E\rightarrow P)$ *and* $(\mathrm{M}E\rightarrow\rightarrow)$*,* q *occurs in* γ*).* [4] *A tree-like* $\mathrm{LM}_{\rightarrow}$ *deduction is called* regular *iff all sequents occurring in it are conclusions of the inferences of maximal priority w.r.t. priority order:*

$(\mathrm{M}A) > (\mathrm{M}I2\rightarrow) > (\mathrm{M}I1\rightarrow) > (\mathrm{M}E\rightarrow P) > (\mathrm{M}E\rightarrow\rightarrow)$.

$$\boxed{(\mathrm{M}A): \qquad \Gamma, p \Longrightarrow p}$$

$$\boxed{(\mathrm{M}I1\rightarrow): \quad \frac{\Gamma,\alpha\Longrightarrow\beta}{\Gamma\Longrightarrow\alpha\rightarrow\beta}\ [(\nexists\gamma):(\alpha\rightarrow\beta)\rightarrow\gamma\in\Gamma]}$$

$$\boxed{(\mathrm{M}I2\rightarrow): \quad \frac{\Gamma,\alpha,\beta\rightarrow\gamma\Longrightarrow\beta}{\Gamma,(\alpha\rightarrow\beta)\rightarrow\gamma\Longrightarrow\alpha\rightarrow\beta}}$$

$$\boxed{(\mathrm{M}E\rightarrow P): \frac{\Gamma,p,\gamma\Longrightarrow q}{\Gamma,p,p\rightarrow\gamma\Longrightarrow q}\,[q\in\mathrm{VAR}(\gamma),p\neq q]}$$

$$\boxed{(\mathrm{M}E\rightarrow\rightarrow): \frac{\Gamma,\alpha,\beta\rightarrow\gamma\Longrightarrow\beta \qquad \Gamma,\gamma\Longrightarrow q}{\Gamma,(\alpha\rightarrow\beta)\rightarrow\gamma\Longrightarrow q}\,[q\in\mathrm{VAR}(\gamma)]}$$

Claim 2 (Hudelmaier) $\mathrm{LM}_{\rightarrow}$ *is sound and complete with respect to minimal propositional logic and tree-like deducibility. Thus any given formula* α *is valid in the minimal logic iff it (i.e. sequent* $\Longrightarrow\alpha$*) is tree-like deducible in* $\mathrm{LM}_{\rightarrow}$*.*

Proof. Easily follows from [6]. ■

In the sequel for any (tree-like or dag-like) deduction ∂ we denote by $h(\partial)$ and $\phi(\partial)$ its *height* (:= maximal thread length in ∂) and *foundation* (:= the total number of pairwise distinct formulas occurring in ∂), respectively. Furthermore for any sequent (in particular, formula) S we denote by $|S|$ the total number of '$\rightarrow$'-occurrences in S and following [6] define the complexity degree $\deg(S)$:

1. $\deg(\Gamma,\alpha\rightarrow\beta\Longrightarrow\alpha) := |\alpha\rightarrow\beta| + \sum_{\xi\in\Gamma}|\xi|$,
2. $\deg(\Gamma\Longrightarrow\alpha) := |\alpha| + \sum_{\xi\in\Gamma}|\xi|$, if $(\nexists\beta):\alpha\rightarrow\beta\in\Gamma$.

Lemma 3

[3] The antecedents, Γ, of our sequents $\Gamma\Longrightarrow\alpha$ are viewed as sets of formulas. Sequents $\Longrightarrow\alpha$, i.e. $\emptyset\Longrightarrow\alpha$, are identified with formulas α.

[4] This slight modification is equivalent to the corresponding subsystem of Hudelmaier's original calculus LG, cf. [6].

1. *Tree-like* $\mathrm{LM}_{\rightarrow}$ *deductions have the semi-subformula property, where semi-subformulas of* $(\alpha \rightarrow \beta) \rightarrow \gamma$ *include* $\beta \rightarrow \gamma$ *along with proper subformulas* $\alpha \rightarrow \beta$, α, β, γ*. I.e. any* β *occurring in a given tree-like* $\mathrm{LM}_{\rightarrow}$ *deduction of* α *is a semi-subformula of* α*.*

2. *If* S' *occurs strictly above* S *in a given tree-like* $\mathrm{LM}_{\rightarrow}$ *deduction* ∂*, then* $\deg(S') < \deg(S)$*.*

3. *The height of any tree-like* $\mathrm{LM}_{\rightarrow}$ *deduction* ∂ *of* S *is linear in* $|S|$*. In particular if* S *is* α*, then* $h(\partial) \leq 3|\alpha|$*.*

4. *The foundation of any tree-like* $\mathrm{LM}_{\rightarrow}$ *deduction* ∂ *of* S *is at most quadratic in* $|S|$*. In particular if* S *is* α*, then* $\phi(\partial) \leq (|\alpha| + 1)^2$*.*

5. *Sequents tree-like deducible in* $\mathrm{LM}_{\rightarrow}$ *are also deducible by regular tree-like deductions.*

Proof. 1: Obvious. Note that $\beta \rightarrow \gamma$ occurring in premises of $(\mathrm{M}I2 \rightarrow)$ and $(\mathrm{M}E \rightarrow\rightarrow)$ are semi-subformulas of $(\alpha \rightarrow \beta) \rightarrow \gamma$ occurring in the conclusions.
2–3: See [6].
4: Let $\mathrm{ssf}(\alpha)$ be the total number of distinct occurrences of semi-subformulas in a given formula α. It is readily seen that $\mathrm{ssf}(-)$ satisfies the following three conditions.

1. $\mathrm{ssf}(p) = 1.$

2. $\mathrm{ssf}(p \rightarrow \alpha) = 2 + \mathrm{ssf}(\alpha).$

3. $\mathrm{ssf}((\alpha \rightarrow \beta) \rightarrow \gamma) = 1 + \mathrm{ssf}(\alpha \rightarrow \beta) + \mathrm{ssf}(\beta \rightarrow \gamma) - \mathrm{ssf}(\beta).$

Moreover 1–3 can be viewed as recursive clauses defining $\mathrm{ssf}(\alpha)$, for any α. Having this we easily arrive at $\mathrm{ssf}(\alpha) \leq (|\alpha| + 1)^2$ (see Appendix A), which by the assertion 1 yields $\phi(\partial) \leq \mathrm{ssf}(\alpha) \leq (|\alpha| + 1)^2$, as required, provided that α (i.e. $\Longrightarrow \alpha$) is the endsequent of ∂.
5: This is easy as, except for the only branching rule $(\mathrm{M}E \rightarrow\rightarrow)$, all other $\mathrm{LM}_{\rightarrow}$ rules are invertible and hence applicable in any order. Note that the regularity assertion is not used in the following proof – it is only relevant in the context of proof search in $\mathrm{LM}_{\rightarrow}$. ■

3.2 $\mathrm{NM}_{\rightarrow}$ and embedding of $\mathrm{LM}_{\rightarrow}$.

Consider system of natural deductions $\mathrm{NM}_{\rightarrow}$ for minimal logic that contains just two rules $(\rightarrow I)$, $(\rightarrow E)$ [9] (here $\rightarrow$ stands for $\supset$).

$$\boxed{\begin{array}{c} [\alpha] \\ \vdots \\ (\rightarrow I) : \dfrac{\beta}{\alpha \rightarrow \beta} \end{array}} \qquad \boxed{(\rightarrow E) : \dfrac{\alpha \quad \alpha \rightarrow \beta}{\beta}}$$

Claim 4 (Prawitz) $\mathrm{NM}_{\to}$ *is sound and complete with respect to minimal propositional logic and tree-like deducibility.*

Proof. See [9]. ■

We'll embed $\mathrm{LM}_{\to}$ into $\mathrm{NM}_{\to}$ following standard pattern *sequent deduction* $\hookrightarrow$ *natural deduction*, where sequent deduction of $\Gamma \Longrightarrow \alpha$ is interpreted as natural deduction of α from open assumptions occurring in Γ. This interpretation is defined as follows by recursion on the input's height (for brevity we don't expose minor assumptions).

$$\boxed{(\mathrm{M}A) : \Gamma, p \Longrightarrow p} \hookrightarrow \boxed{p}$$

$$\boxed{\begin{array}{c}(\mathrm{M}I1 \to)\ :\ \dfrac{\Gamma, \alpha \Longrightarrow \beta}{\Gamma \Longrightarrow \alpha \to \beta} \\ [(\nexists \gamma) : (\alpha \to \beta) \to \gamma \in \Gamma]\end{array}} \hookrightarrow \boxed{\begin{array}{c}[\alpha] \\ \Downarrow \\ \dfrac{\beta}{\alpha \to \beta}\ (\to I)\end{array}}$$

(*discharging premise-assumption* α)

$$\boxed{(\mathrm{M}I2 \to) :\ \dfrac{\Gamma, \alpha, \beta \to \gamma \Longrightarrow \beta}{\Gamma, (\alpha \to \beta) \to \gamma \Longrightarrow \alpha \to \beta}} \hookrightarrow$$

$$\boxed{\begin{array}{ccc} & & \dfrac{\dfrac{\dfrac{[\beta]^2}{\alpha \to \beta}\ (\to I) \qquad (\alpha \to \beta) \to \gamma}{\gamma}\ (\to E)}{\beta \to \gamma^{[2]}}\ (\to I) \\ [\alpha]^1 & & \\ \Downarrow & & \Downarrow \\ & \searrow \quad \swarrow & \\ & \dfrac{\beta}{\alpha \to \beta^{[1]}}\ (\to I) & \end{array}}$$

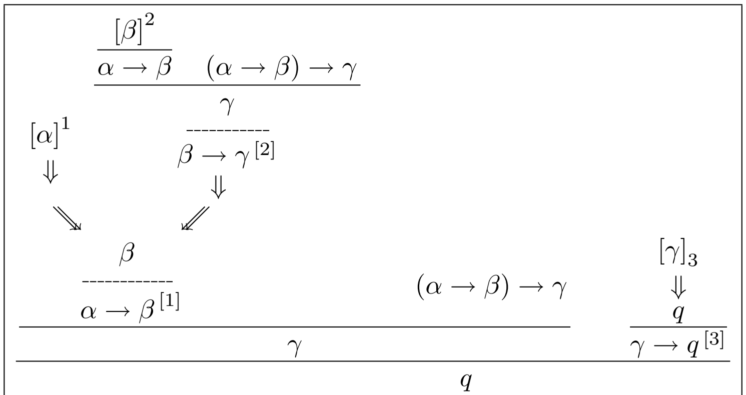
(*discharging/deducing premise-assumptions* $\alpha, \beta \to \gamma$)

$$\boxed{\begin{array}{c}(\mathrm{M}E \to P) :\ \dfrac{\Gamma, p, \gamma \Longrightarrow q}{\Gamma, p, p \to \gamma \Longrightarrow q} \\ [q \in \mathrm{VAR}(\gamma), p \neq q]\end{array}} \hookrightarrow$$

$$\boxed{\begin{array}{ccc} p & & \dfrac{p \quad p \to \gamma}{\gamma}\ (\to E) \\ \Downarrow & & \Downarrow \\ & \searrow \quad \swarrow & \\ & q & \end{array}}$$

(*deducing premise-assumption* γ)

$$\boxed{(\mathrm{M}E \to\to) : \frac{\Gamma, \alpha, \beta \to \gamma \Longrightarrow \beta \qquad \Gamma, \gamma \Longrightarrow q}{\Gamma, (\alpha \to \beta) \to \gamma \Longrightarrow q} \left[q \in \mathrm{VAR}(\gamma) \right]} \hookrightarrow$$



$$\left(\begin{array}{c} \textit{discharging/deducing premise-assumptions } \alpha, \beta \to \gamma, \gamma \\ \textit{while for brevity dropping names of the rules} \end{array} \right)$$

Lemma 5 *There exists a recursive operator* $\digamma$ *transforming any given tree-like* $\mathrm{LM}_{\to}$ *deduction* ∂ *of* $\Gamma \Longrightarrow \alpha$ *into a tree-like* $\mathrm{NM}_{\to}$ *deduction* $\digamma(\partial)$ *with endformula* α *and assumptions occurring in* Γ. *Moreover* ∂ *and* $\digamma(\partial)$ *share the semi-subformula property, linearity of the height and polynomial upper bounds on the foundation. In particular if* $\Gamma = \emptyset$, *then* $\digamma(\partial)$ *is a* $\mathrm{NM}_{\to}$ *proof of* α *such that* $h(\digamma(\partial)) \leq 18|\alpha|$ *and* $\phi(\digamma(\partial)) < (|\alpha|+1)^2(|\alpha|+2)$.

Proof. $\digamma(\partial)$ is defined by recursion on $h(\partial)$ using embedding clauses shown above. Each clause increases the height at most by 6 (in the case $(\mathrm{M}E \to\to)$), which yields $h(\digamma(\partial)) \leq 6 \cdot h(\partial) \leq 18|\alpha|$ according to Lemma 3 (3). By the same token, formulas occurring in $\digamma(\partial)$ include the ones occurring in ∂ together with possibly new formulas $\gamma \to q$ (with old γ and q) shown on the right-hand side in the case $(\mathrm{M}E \to\to)$. Clearly there are $< \phi(\partial)$ and $\leq |\alpha|+1$ such γ and q, respectively. Hence by Lemma 3 (4) we arrive at $\phi(\digamma(\partial)) < (|\alpha|+1)^2 + (|\alpha|+1)^2(|\alpha|+1) = (|\alpha|+1)^2(|\alpha|+2)$, as required. ■

3.3 Dag-like deducibility in $\mathrm{NM}_{\to}$.

We'll consider dag-like natural deductions, instead of familiar tree-like ones. Recall that *'dag'* stands for *directed acyclic graph* (edges directed upwards). The main difference between tree-like and dag-like natural deductions is caused by the art of discharging, as the following example shows.

Example 6 *Consider a dag-like natural deduction* $\partial =$

$$(\to E)\ \dfrac{(\to E)\ \dfrac{\overbrace{\vdots}^{\Gamma} \atop \beta\to\alpha \qquad \dfrac{[\alpha] \qquad \alpha\to\beta}{\beta}\ (\to E)}{\alpha \qquad\qquad \alpha\to\beta}\ (\to I)}{\beta}$$

in which the right-hand side premise of second $(\to \mathrm{E})$ *coincides with* $(\to \mathrm{I})$ *premise* β*. Note that the assumption* α *above* β *is discharged by this* $(\to \mathrm{I})$*. However, we can only infer that* ∂ *deduces* β *from* $\Gamma \cup \{\alpha, \alpha \to \beta\}$*, instead of expected* $\Gamma \cup \{\alpha \to \beta\}$*, which leaves the option* $\Gamma \cup \{\alpha \to \beta\} \nvdash \beta$ *open, if* $\alpha \notin \Gamma$*. So the assumption* α *is actually open in the whole* ∂ *due to a* $(\to \mathrm{I})$*-free thread* $\alpha, \beta, \alpha, \beta$*. This becomes obvious if we replace* ∂ *by the "unfolded" tree-like* $\partial_{\mathrm{U}} :=$

$$(\to E)\ \dfrac{\dfrac{\dfrac{\alpha \qquad \alpha\to\beta}{\beta}\ (\to E) \qquad \overbrace{\vdots}^{\Gamma} \atop \beta\to\alpha}{\alpha}\ (\to E) \qquad \dfrac{\dfrac{[\alpha] \qquad \alpha\to\beta}{\beta}\ (\to E)}{\alpha\to\beta}\ (\to I)}{\beta}$$

Keeping this in mind we'll say that in a dag-like natural deduction ∂, a given leaf u labeled with formula α is an *open* (or *undischarged*) *assumption-node*, and α is an *open* (or *undischarged*) *assumption*, iff there exists a thread θ connecting u with the root such that no $w \in \theta$ is the $(\to \mathrm{I})$ conclusion labeled with formula $\alpha \to \beta$. Other leaves are called *closed* (or *discharged*) *assumption-nodes*. Note that the corresponding condition '*u is open* (resp. *closed*) *in* ∂' belongs merely to **NP** (resp. **coNP**), unless ∂ is a tree-like deduction, in which case both conditions are in **P**, as desired. We'll overcome this trouble by a suitable modification of the notion of local correctness that includes generalized discharging function $\ell^{\mathrm{d}} : \mathrm{V}(D) \times \mathrm{A}(D) \to \{0, 1\}$ (see below).

Definition 7 *Consider quadruples* $\mathcal{D} = \left\langle D, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{d} \right\rangle$ *where* D *are finite mo-noedge* [5] *rooted dags and* $\ell^{\mathrm{F}} : \mathrm{V}(D) \to \mathrm{F}$*,* $\ell^{\mathrm{R}} : \mathrm{V}(D) \to \mathrm{R} \cup \{\emptyset\}$*,* $\ell^{d} : \mathrm{V}(D) \times \mathrm{A}(D) \to \{0, 1\}$ *vertex-labeling functions. Here* $\mathrm{V}(D)$ *are the vertices (= nodes) of* D*,* F *and* R *the formulas and names of the rules of* $\mathrm{NM}_{\to}$*, respectively, while* $\mathrm{A}(D) = \{\alpha : \alpha \to \beta \in \mathrm{Rng}\,(\ell^{\mathrm{F}})\}$*.* $\mathcal{D} = \left\langle D, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{d} \right\rangle$ *is called* locally correct *iff the following conditions 1–7 hold for all* $u \in \mathrm{V}(D)$ *and* $\alpha \in \mathrm{F}(D)$*, where* $\overrightarrow{\deg}\,(u)$ *(resp.* $\overleftarrow{\deg}\,(u)$*) is straight (resp. inverse) branching degree of* u [6] *and* $\ell^{\mathrm{RF}}\,(u) = (\to I)_{\alpha}$ *stands for* $\ell^{\mathrm{R}}\,(u) = (\to I) \wedge \ell^{\mathrm{F}}\,(u) = \alpha \to \ell^{\mathrm{F}}\left(u^{(1)}\right)$*.*

1. *Except for the leaves, every* u *has one or two (ordered) children* $u^{(1)}$ *or* $u^{(1)}, u^{(2)}$*, respectively, and* $\overleftarrow{\deg}\,(u)$ *(ordered) parents* $u_{(1)}, \cdots, u_{\left(\overleftarrow{\deg}(u)\right)}$*.*

   Thus for any $u \in \mathrm{V}(D)$ *we have* $2 \geq \overrightarrow{\deg}\,(u) \geq 0 \leq \overleftarrow{\deg}\,(u) < |\mathrm{V}\,(D)|$*.*

[5] That is, in D, any two distinct vertices can be connected by at most one (directed) edge.

[6] That is, $\overrightarrow{\deg}\,(u)$ (resp. $\overleftarrow{\deg}\,(u)$) is the total number of targets with source u (resp. total number of sources with target u).

2. *If* $\overrightarrow{\deg}(u) = 0$, *i.e.* u *is a leaf, then* $\ell^{\mathrm{R}}(u) = \emptyset$.

3. *If* $\overrightarrow{\deg}(u) = 1$, *then* $(\exists \alpha \in \mathrm{A}(D))\, \ell^{\mathrm{RF}}(u) = (\to I)_{\alpha}$.

4. *If* $\overrightarrow{\deg}(u) = 2$, *then* $\ell^{\mathrm{R}}(u) = (\to E)$ *and* $\ell^{\mathrm{F}}\left(u^{(2)}\right) = \ell^{\mathrm{F}}\left(u^{(1)}\right) \to \ell^{\mathrm{F}}(u)$.

5. *If* $\overleftarrow{\deg}(u) = 0$, *i.e.* u *is the root, then* $\ell^{d}(u, \alpha) = 1$ *iff* $\ell^{\mathrm{RF}}(u) = (\to I)_{\alpha}$.

6. *If* $\ell^{\mathrm{RF}}(u) = (\to I)_{\alpha}$, *then* $\ell^{d}(u, \alpha) = 1$.

7. *If* $\overleftarrow{\deg}(u) > 0$ *and* $\ell^{\mathrm{RF}}(u) \neq (\to I)_{\alpha}$, *then* $\ell^{d}(u, \alpha) = \prod_{i=1}^{\overleftarrow{\deg}(u)} \ell^{d}\left(u_{(i)}, \alpha\right)$.

Call $\Gamma_{\mathcal{D}} := \left\{\ell^{\mathrm{F}}(u) : u \in \mathrm{V}(D) \wedge 0 = \overrightarrow{\deg}(u) = \ell^{d}(u, \ell^{\mathrm{F}}(u))\right\}$ *the set of* open *(or* undischarged*)* assumptions, *in* $\mathcal{D}$. *Denote by* $\mathbb{D}_{\to}$ *the set of locally correct quadruples* $\mathcal{D}$. *Now any* $\mathcal{D} = \left\langle D, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{d} \right\rangle \in \mathbb{D}_{\to}$ *is called an* encoded dag-like natural deduction of $\ell^{\mathrm{F}}(\mathrm{root}(D))$ from $\Gamma_{\mathcal{D}}$. *In particular, if* $\Gamma_{\mathcal{D}} = \emptyset$, *then* $\mathcal{D}$ *is called an* encoded dag-like proof of $\ell^{\mathrm{F}}(\mathrm{root}(D))$, *in* $\mathrm{NM}_{\to}$.

Lemma 8 *There is an isomorphism between plain (i.e. unencoded) and encoded dag-like natural deductions (in particular, proofs), in* $\mathrm{NM}_{\to}$. *Thus the open assumptions in a given plain deduction* ∂ *(see above) are the elements of* $\Gamma_{\mathcal{D}}$ *in the correlated encoded deduction* $\mathcal{D}$.

Proof. A given plain dag-like natural deduction ∂ is a dag D whose nodes u are labeled with formulas α_u and (names of) rules R_u assigned to the conclusions (so the leaves have no rule labels). To obtain the corresponding encoded dag-like deduction $\mathcal{D} = \left\langle D, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{\mathrm{d}} \right\rangle \in \mathbb{D}_{\to}$ we let $\ell^{\mathrm{F}}(u) := \alpha_u$ and $\ell^{\mathrm{R}}(u) := R_u$ if $\overrightarrow{\deg}(u) > 0$, else $\emptyset$. The ℓ^{d}-labels are uniquely determined by recursion on $h(D)$ using clauses 5–7 of Definition 7. Other conditions of the local correctness are easily verified. Also note that the following equation holds.

$$\ell^{\mathrm{d}}(u, \alpha) = \begin{cases} 0 & \text{if} \quad (\exists \, thread \ \theta \in [u, root(D)])\, (\forall w \in \theta)\, \ell^{\mathrm{RF}}(w) \neq (\to I)_{\alpha}, \\ 1 & \text{else.} \end{cases} \tag{$*$}$$

Consequently the open (resp. closed) assumption-nodes u of ∂ (see above) are the ones satisfying $\overrightarrow{\deg}(u) = 0$ and $\ell^{\mathrm{d}}(u, \ell^{\mathrm{F}}(u)) = 0$ (resp. $\ell^{\mathrm{d}}(u, \ell^{\mathrm{F}}(u)) = 1$). Hence the open assumptions in ∂ are just the elements of $\Gamma_{\mathcal{D}}$. The other direction is analogous. For any $\mathcal{D} = \left\langle D, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{\mathrm{d}} \right\rangle \in \mathbb{D}_{\to}$ we define ∂ by $\alpha_u := \ell^{\mathrm{F}}(u)$ and $R_u := \ell^{\mathrm{R}}(u)$ and use 5–7 to prove $(*)$ by induction on $h(D)$. ■

Example 9 *Consider plain deductions* ∂ *and* ∂_{U} *exposed in Example 6, where for brevity we assume that in* ∂ *the subdag over* $(\to \mathrm{E})$ *premise* $\beta \to \alpha$ *is a tree. Denote by* u, v, w *the root and the nodes of* $(\to \mathrm{E})$ *premises* α *and* $\alpha \to \beta$, *respectively, in both* ∂ *and* ∂_{U}. *Furthermore let* x *and* y *be the nodes of* $\beta \to \alpha$

and $(\to \mathrm{I})$ *premise* β*, respectively, in both* ∂ *and* ∂_{U}*. Finally, let* z *be the node of the conclusion* β *of the top-left* $(\to \mathrm{E})$ *in* ∂_{U}*. This yields* $\partial =$

$$\boxed{(\to \mathrm{E})\,\dfrac{(\to \mathrm{E})\,\dfrac{\overbrace{\vdots}^{\Gamma} \qquad x : \beta \to \alpha \qquad \dfrac{[\alpha]^{1} \qquad \alpha \to \beta}{y : \beta}\,(\to \mathrm{E})}{v : \alpha \qquad\qquad w : \alpha \to \beta^{[1]}}\,(\to \mathrm{I})}{u : \beta}}$$

and $\partial_{\mathrm{U}} =$

$$\boxed{(\to \mathrm{E})\,\dfrac{\dfrac{\dfrac{\alpha \qquad \alpha \to \beta}{z : \beta}\,(\to \mathrm{E}) \qquad \overbrace{\vdots}^{\Gamma} \; x : \beta \to \alpha}{v : \alpha}\,(\to \mathrm{E}) \qquad \dfrac{\dfrac{[\alpha]_{1} \qquad \alpha \to \beta}{y : \beta}\,(\to \mathrm{E})}{w : \alpha \to \beta^{[1]}}\,(\to \mathrm{I})}{u : \beta}}$$

So $v = u^{(1)}$*,* $w = u^{(2)}$*,* $x = v^{(2)}$ *and* $y = w^{(1)}$ *in both* ∂ *and* ∂_{U}*. Moreover* $y = v^{(1)}$ *in* ∂ *and* $z = v^{(1)}$ *in* ∂_{U}*. On the other hand* $u = v_{(1)} = w_{(1)}$*,* $v = x_{(1)}$ *in both* ∂ *and* ∂_{U}*,* $v = y_{(1)}$*,* $w = y_{(2)}$ *in* ∂ *and* $v = z_{(1)}$*,* $w = y_{(1)}$ *in* ∂_{U}*. To get the encoded deductions* $\mathcal{D} = \left\langle D, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{d} \right\rangle$ *and* $\mathcal{D}_{\mathrm{U}} = \left\langle D_{\mathrm{U}}, \ell^{\mathrm{F}}_{\mathrm{U}}, \ell^{\mathrm{R}}_{\mathrm{U}}, \ell^{d}_{\mathrm{U}} \right\rangle$ *we add:*

- $\ell^{\mathrm{F}}(u) = \ell^{\mathrm{F}}_{\mathrm{U}}(u) = \beta$*,* $\ell^{\mathrm{R}}(u) = \ell^{\mathrm{R}}_{\mathrm{U}}(u) = (\to \mathrm{E})$*,*
- $\ell^{\mathrm{F}}(v) = \ell^{\mathrm{F}}_{\mathrm{U}}(v) = \alpha$*,* $\ell^{\mathrm{R}}(v) = \ell^{\mathrm{R}}_{\mathrm{U}}(v) = (\to \mathrm{E})$*,*
- $\ell^{\mathrm{F}}(w) = \ell^{\mathrm{F}}_{\mathrm{U}}(w) = \alpha \to \beta$*,* $\ell^{\mathrm{R}}(w) = \ell^{\mathrm{R}}_{\mathrm{U}}(w) = (\to \mathrm{I})$*,*

etc., and define $\ell^{d}(-,-)$ *and* $\ell^{d}_{\mathrm{U}}(-,-)$ *from the bottom up, successively:*

- $\ell^{d}(u,\xi) = \ell^{d}_{\mathrm{U}}(u,\xi) = 0$*, for all* ξ *(by clause 5 of Definition 7),*
- $\ell^{d}(v,\xi) = \ell^{d}_{\mathrm{U}}(v,\xi) = 0$*, for all* ξ *(by clause 7 of Definition 7),*
- $\ell^{d}(w,\xi) = \ell^{d}_{\mathrm{U}}(w,\xi) = 1$*, if* $\xi = \alpha$*, else* 0 *(by clauses 6, 7 of Definition 7),*
- $\ell^{d}(x,\xi) = \ell^{d}_{\mathrm{U}}(x,\xi) = 0$*, for all* ξ *(by clause 7 of Definition 7),*
- $\ell^{d}(y,\xi) = 0 \cdot 1 = 0$*, if* $\xi = \alpha$*, else* 0*; hence* $\ell^{d}(y,\xi) = 0$ *for all* ξ *(by clause 7 of Definition 7),*
- $\ell^{d}_{\mathrm{U}}(y,\xi) = 1$*, if* $\xi = \alpha$*, else* 0 *(by clause 7 of Definition 7),*
- $\ell^{d}_{\mathrm{U}}(z,\xi) = 0$*, for all* ξ *(by clause 7 of Definition 7),*
- $\ell^{d}\left(y^{(1)},\xi\right) = \ell^{d}\left(y^{(2)},\xi\right) = 0$*, for all* ξ *(by clause 7 of Definition 7),*
- $\ell^{d}_{\mathrm{U}}\left(y^{(1)},\xi\right) = \ell^{d}_{\mathrm{U}}\left(y^{(2)},\xi\right) = 1$*, if* $\xi = \alpha$*, else* 0 *(by clause 7 of Definition 7),*

- $\ell_{\mathrm{U}}^{d}\left(z^{(1)},\xi\right)=\ell_{\mathrm{U}}^{d}\left(z^{(2)},\xi\right)=0$, *for all* ξ *(by clause 7 of Definition 7),*

etc., which in particular yields $\ell^{d}\left(y^{(1)},\alpha\right)=\ell_{\mathrm{U}}^{d}\left(z^{(1)},\alpha\right)=0$, *and hence* $\alpha\in\Gamma_{\mathcal{D}}\cap\Gamma_{\mathcal{D}_{\mathrm{U}}}$. *A more general corollary reads* $\Gamma_{\mathcal{D}}=\Gamma_{\mathcal{D}_{\mathrm{U}}}=\Gamma\cup\{\alpha,\alpha\rightarrow\beta\}$. *Since* $\mathcal{D}$ *and* $\mathcal{D}_{\mathrm{U}}$ *are just encoded versions of* ∂ *and* ∂_{U}*, respectively, this confirms our initial observation that dag-like* ∂ *and its unfolded tree-like counterpart* ∂_{U} *both deduce* β *from the set of assumptions* $\Gamma\cup\{\alpha,\alpha\rightarrow\beta\}$*, but not* $\Gamma\cup\{\alpha\rightarrow\beta\}$.

Unless stated otherwise, we'll identify plain and encoded notions of deducibility (provability) and use abbreviations $\mathrm{V}(\mathcal{D})$, $h(\mathcal{D})$ and $\phi(\mathcal{D})$ for the vertices, height and foundation of plain dag-like representations ∂ of $\mathcal{D}$. In the sequel we set $root(\mathcal{D}):=root(D)$ and $|\mathcal{D}|:=|\mathrm{V}(\mathcal{D})|$ (= the size of $\mathcal{D}$). Note that standard tree-like deducibility is a special case of the dag-like one (whether plain or encoded), assuming that D is a tree. In this case we'd rather rename D to T.

Lemma 10 $\mathrm{NM}_{\rightarrow}$ *is sound and complete with respect to minimal propositional logic and encoded dag-like deducibility (in particular, provability). Moreover for any given quadruple* $\mathcal{D}=\left\langle D,\ell^{\mathrm{F}},\ell^{\mathrm{R}},\ell^{d}\right\rangle$*, the condition '*$\mathcal{D}$ is encoded dag-like proof*' is decidable in* $|\mathcal{D}|$*-polynomial time.*

Proof. The completeness follows immediately from Claim 4, since tree-like deducibility is a special case of the dag-like one. The soundness can be proved straightforward, but for the sake of coherence we'll also reduce it to Claim 4. To this end we observe that every encoded dag-like $\mathrm{NM}_{\rightarrow}$ proof $\mathcal{D}$ (actually any deduction) is convertible by natural downward *unfolding* to the uniquely determined (modulo isomorphism) encoded tree-like $\mathrm{NM}_{\rightarrow}$ proof (deduction) $\mathcal{D}_{\mathrm{U}}$ of the same conclusion (from the same open assumptions, in the general deduction case). So let $\mathcal{D}=\left\langle D,\ell^{\mathrm{F}},\ell^{\mathrm{R}},\ell^{\mathrm{d}}\right\rangle\in\mathbb{D}_{\rightarrow}$. A desired unfolded encoded tree-like $\mathrm{NM}_{\rightarrow}$ proof of $root(D)$ is obtained by setting $\mathcal{D}_{\mathrm{U}}:=\widetilde{\mathcal{D}}_{h(D)-1}$ where $\widetilde{\mathcal{D}}_{n}=\left\langle D_{n},\ell_{n}^{\mathrm{F}},\ell_{n}^{\mathrm{R}},\ell_{n}^{\mathrm{d}}\right\rangle\in\mathbb{D}_{\rightarrow}$ is defined as follows by recursion on $n\leq h(D)-2$. Below, for any $n\leq h(D)$ we let $L_{n}:=\{x\in\mathrm{V}(D):h(x)=n\}$, where $h(x)$ is the height of x, i.e. maximal distance between x and $root(D)$, in D.

1. Basis: $\widetilde{\mathcal{D}}_{0}:=\mathcal{D}$.

2. Recursion step $n\leq h(D)-2$. Consider $\widetilde{\mathcal{D}}_{n}=\left\langle D_{n},\ell_{n}^{\mathrm{F}},\ell_{n}^{\mathrm{R}},\ell_{n}^{\mathrm{d}}\right\rangle\in\mathbb{D}_{\rightarrow}$. By the construction (induction hypothesis) we have $L_{h(D)-n}\subset\mathrm{V}(D_{n})$, while all $u\in L_{h(D)-n}$ are roots of pairwise disjoint unfolded upper subtrees $V_{\succeq u}=\{x\in\mathrm{V}(D_{n}):u\preceq x\}$, where $u\preceq x\Leftrightarrow$ '*x occurs above, or coincides with,* u'. For every $u\in L_{h(D)-n}$ let $\overleftarrow{u}=\left\{u_{1}^{\prime},\cdots,u_{\overleftarrow{\deg}(u)}^{\prime}\right\}$ be the set of $\overleftarrow{\deg}(u)$ new distinct vertices, if $\overleftarrow{\deg}(u)>1$, else $\overleftarrow{u}:=\{u\}$. Moreover let $\{V_{\succeq u^{\prime}}:u^{\prime}\in\overleftarrow{u}\}$ be the set of pairwise disjoint copies of $V_{\succeq u}$. Also let $L_{h(D)-n}^{+}:=\bigcup_{u\in L_{h(D)-n}}\overleftarrow{u}$. Now $\widetilde{\mathcal{D}}_{n+1}=\left\langle D_{n+1},\ell_{n+1}^{\mathrm{F}},\ell_{n+1}^{\mathrm{R}},\ell_{n+1}^{\mathrm{d}}\right\rangle$ arises from $\widetilde{\mathcal{D}}_{n}$ by extending $L_{h(D)-n}$ to $L_{h(D)-n}^{+}$ whose new vertices $u^{\prime}\in\overleftarrow{u}$ are regarded as roots of the corresponding upper subtrees $V_{\succeq u^{\prime}}$, while new edges targeting these $u^{\prime}$ are copies of the corresponding old edges targeting u. ℓ_{n+1}^{F}, ℓ_{n+1}^{R} and ℓ_{n+1}^{d} are inherited analogously.

To put it more precisely in the latter case, if $1 \leq i \leq \overleftarrow{\deg}(u) > 1$, then for any $\alpha \in \mathrm{A}(D_n)$ we put $\ell^{\mathrm{d}}_{n+1}(u'_i, \alpha) := \ell^{\mathrm{d}}_n(u_{(i)}, \alpha)$ and use clauses 6, 7 of Definition 7 to generate $\ell^{\mathrm{d}}_{n+1}(x, \alpha)$ for all $x \in V_{\succeq u'_i} \setminus \{u'_i\}$. Note that our definition yields

$$\ell^{\mathrm{F}}_n(u) = \ell^{\mathrm{F}}_{n+1}(u'_i), \ \ell^{\mathrm{R}}_n(u) = \ell^{\mathrm{R}}_{n+1}(u'_i) \text{ and } \ell^{\mathrm{d}}_n(u, \alpha) = \prod_{i=1}^{\overleftarrow{\deg}(u)} \ell^{\mathrm{d}}_{n+1}(u'_i, \alpha) \qquad (**)$$

It is readily seen that $\widetilde{\mathcal{D}}_n \hookrightarrow \widetilde{\mathcal{D}}_{n+1}$ preserves the local correctness. Hence $\widetilde{\mathcal{D}}_{n+1} \in \mathbb{D}_{\rightarrow}$. Moreover $\ell^{\mathrm{F}}_{n+1}(\mathit{root}(\mathcal{D}_{n+1})) = \ell^{\mathrm{F}}_n(\mathit{root}(\mathcal{D}_n)) = \ell^{\mathrm{F}}(\mathit{root}(\mathcal{D}))$ and $\ell^{\mathrm{R}}_{n+1}(\mathit{root}(\mathcal{D}_{n+1})) = \ell^{\mathrm{R}}_n(\mathit{root}(\mathcal{D}_n)) = \ell^{\mathrm{R}}(\mathit{root}(\mathcal{D}))$.

3. Having arrived at $n = h(D) - 1$ observe that $D_{h(D)-1}$ is a tree, since D is at most binary-branching monoedge dag and $\{x \in \mathrm{V}(D_{h(D)-1}) : h(x) > 1\}$ is tree-like ordered by the induction hypothesis. Hence $\mathcal{D}_{\mathrm{U}} = \widetilde{\mathcal{D}}_{h(D)-1}$ is an encoded tree-like natural deduction of $\ell^{\mathrm{F}}(\mathit{root}(\mathcal{D}))$. Moreover $\mathcal{D}$ and $\mathcal{D}_{\mathrm{U}}$ both have the same open assumptions. To recognize this we use equation $(*)$ (see proof of Lemma 8). Namely, let u be any open assumption-node labeled with α and $\theta = [u = w_0, w_1, \cdots, w_m = \mathit{root}(D)]$, $w_{k+1} = (w_k)_{(i)}$, the correlated "open" thread, in $\mathcal{D}$. So for all $k \leq m$ we have $\ell^{\mathrm{d}}(w_k, \alpha) = 0$. Now observe that θ determines an analogous "open" thread $\theta' = [u' = w'_0, w'_1, \cdots, w'_m = w_m]$, $w'_{k+1} = (w'_k)_{(i)}$, $\ell^{\mathrm{d}}_{\mathrm{U}}(w'_k, \alpha) = 0$, in $\mathcal{D}_{\mathrm{U}}$, in which u' is an assumption-node also labeled with α. This conclusion follows from $(**)$ by the recursive definition of $\mathcal{D}_{\mathrm{U}}$. Consequently α is an open assumption also in $\mathcal{D}_{\mathrm{U}}$. The other direction follows by the same token. Summing up, we arrive at $\Gamma_{\mathcal{D}_{\mathrm{U}}} = \Gamma_{\mathcal{D}}$, and hence $\mathcal{D}_{\mathrm{U}}$ is an encoded tree-like natural deduction of $\ell^{\mathrm{F}}(\mathit{root}(\mathcal{D}))$ from $\Gamma_{\mathcal{D}}$. So if $\Gamma_{\mathcal{D}} = \emptyset$, i.e. $\mathcal{D}$ is encoded dag-like $\mathrm{NM}_{\rightarrow}$ proof of $\ell^{\mathrm{F}}(\mathit{root}(\mathcal{D}))$, then $\mathcal{D}_{\mathrm{U}}$ is a desired encoded tree-like $\mathrm{NM}_{\rightarrow}$ proof of $\ell^{\mathrm{F}}(\mathit{root}(\mathcal{D}))$. Hence the validity of $\ell^{\mathrm{F}}(\mathit{root}(\mathcal{D}))$ in minimal logic follows from Claim 4. This completes the soundness proof.

The last assertion of Lemma easily follows from Definition 7 and standard encoding of property to be a dag (that includes all parameters involved like $\overrightarrow{\deg}(u)$, $\overleftarrow{\deg}(u)$, $u^{(i)}$, $u_{(i)}$, etc.). ■

3.4 Horizontal dag-like compression in $\mathrm{NM}_{\rightarrow}$.

Horizontal dag-like compression of a given tree-like $\mathrm{NM}_{\rightarrow}$ deduction is thought of as the ultimate inverse of the above unfolding. It is obtained by iteration of *horizontal collapsing* of pairs of distinct vertices labeled by equal formulas.

Definition 11 (*horizontal collapsing*) *Let $\mathcal{D} = \langle D, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{d} \rangle \in \mathbb{D}_{\rightarrow}$ and $u \neq w \in \mathrm{V}(D)$ such that $V_{\succeq u} = \{x \in \mathrm{V}(D) : u \preceq x\}$, $V_{\succeq w} = \{x \in \mathrm{V}(D) : w \preceq x\}$ are disjoint (sub)trees with $\ell^{\mathrm{F}}(u) = \ell^{\mathrm{F}}(w)$. Moreover w.l.o.g. we'll assume that either $\ell^{\mathrm{R}}(u) = (\rightarrow I)$, or else $\ell^{\mathrm{R}}(u) \neq (\rightarrow I) \neq \ell^{\mathrm{R}}(w)$ and $h(V_{\succeq u}) \leq h(V_{\succeq w})$.* [7] *Now $\mathcal{D}_{u \hookleftarrow w} = \langle D_{u \hookleftarrow w}, \ell^{\mathrm{F}}_{u \hookleftarrow w}, \ell^{\mathrm{R}}_{u \hookleftarrow w}, \ell^{d}_{u \hookleftarrow w} \rangle \in \mathbb{D}_{\rightarrow}$ is stipulated as follows.*

[7] The latter condition is not crucial, but it can reduce the size of the collapsed deduction.

1. $D_{u\hookleftarrow w}$ is a subdag of D with vertices $\mathrm{V}(D_{u\hookleftarrow w}) := \mathrm{V}(D) \setminus V_{\succeq w}$ and edges $\mathrm{E}(D_{u\hookleftarrow w}) := \mathrm{E}(D)\downarrow_{\mathrm{V}(D_{u\hookleftarrow w})^2} \cup \{\langle x,u\rangle : x \in \mathrm{V}(D_{u\hookleftarrow w})\}_{\langle x,w\rangle \in \mathrm{E}(D)}$. [8]

2. $\ell^{\mathrm{F}}_{u\hookleftarrow w} := \ell^{\mathrm{F}}\downarrow_{\mathrm{V}(D_{u\hookleftarrow w})}$ and $\ell^{\mathrm{R}}_{u\hookleftarrow w} := \ell^{\mathrm{R}}\downarrow_{\mathrm{V}(D_{u\hookleftarrow w})}$.

3. $\ell^{d}_{u\hookleftarrow w}$ is defined by cases as follows, for all $x \in \mathrm{V}(D_{u\hookleftarrow w})$ and $\alpha \in \mathrm{A}(D_{u\hookleftarrow w})$.

(a) If $\ell^{\mathrm{R}}(u) = (\to I)$, then $\ell^{d}_{u\hookleftarrow w}(x,\alpha) := \ell^{d}(x,\alpha)$.

(b) If $\ell^{\mathrm{R}}(u) \neq (\to I) \neq \ell^{\mathrm{R}}(w)$, then:

i. $\ell^{d}_{u\hookleftarrow w}(u,\alpha) := \ell^{d}(u,\alpha)\cdot\ell^{d}(w,\alpha)$, which generates $\ell^{d}_{u\hookleftarrow w}(x,\alpha)$ also for all $x \in h(V_{\succeq u}) \setminus \{u\}$ by clauses 6, 7 of Definition 7,

ii. $\ell^{d}_{u\hookleftarrow w}(x,\alpha) := \ell^{d}(x,\alpha)$, if $x \in \mathrm{V}(D_{u\hookleftarrow w}) \setminus V_{\succeq u}$.

The operation $\mathcal{D} \hookrightarrow \mathcal{D}_{u\hookleftarrow w}$ is called horizontal collapsing. [9]

Conclusion 12 *It is readily seen that $\mathcal{D}_{u\hookleftarrow w} \in \mathbb{D}_{\to}$ and $\ell^{\mathrm{F}}_{u\hookleftarrow w}(\mathrm{root}(\mathcal{D}_{u\hookleftarrow w})) = \ell^{\mathrm{F}}(\mathrm{root}(\mathcal{D}))$ with $\ell^{\mathrm{R}}_{u\hookleftarrow w}(\mathrm{root}(\mathcal{D}_{u\hookleftarrow w})) = \ell^{\mathrm{R}}(\mathrm{root}(\mathcal{D}))$. Moreover $\Gamma_{\mathcal{D}_{u\hookleftarrow w}} \subseteq \Gamma_{\mathcal{D}}$ as, by clause 3, for every leaf $y \in \mathrm{V}(D_{u\hookleftarrow w})$ satisfying $\ell^{d}_{u\hookleftarrow w}(y, \ell^{\mathrm{F}}_{u\hookleftarrow w}(y)) = 0$ there exists a leaf $y' \in \mathrm{V}(D)$ with $\ell^{\mathrm{F}}(y') = \ell^{\mathrm{F}}_{u\hookleftarrow w}(y)$ and $\ell^{d}(y', \ell^{\mathrm{F}}(y')) = 0$. Also note that $|\mathcal{D}_{u\hookleftarrow w}| < |\mathcal{D}|$.*

Definition 13 (*horizontal tree-to-dag compression*) *For any given tree-like $\mathcal{D} = \langle T, \ell^{\mathrm{F}}, \ell^{\mathrm{R}}, \ell^{d}\rangle \in \mathbb{D}_{\to}$ its* horizontal tree-to-dag compression *$\mathcal{D}_{\mathrm{C}} = \langle D_{\mathrm{C}}, \ell^{\mathrm{F}}_{\mathrm{C}}, \ell^{\mathrm{R}}_{\mathrm{C}}, \ell^{d}_{\mathrm{C}}\rangle \in \mathbb{D}_{\to}$ is obtained by bottom-up iteration of horizontal collapsing so long as possible, starting with $\mathcal{D}$. More precisely, we set $\mathcal{D}_{\mathrm{C}} := \widehat{\mathcal{Q}}_{h(T)+1}$ and define $\widehat{\mathcal{D}}_{n}$ as follows by recursion on $n \leq h(T)$.*

1. Basis: $\widehat{\mathcal{D}}_0 := \widehat{\mathcal{D}}_1 := \mathcal{D}$.

2. Recursion step n. Let $0 < n \leq h(T)$ and $\widehat{\mathcal{D}}_n = \langle D_n, \ell^{\mathrm{F}}_n, \ell^{\mathrm{R}}_n, \ell^{d}_n\rangle$. By the construction (induction hypothesis), $L_n = \{x \in \mathrm{V}(D_n) : h(x) = n\} \subset \mathrm{V}(T)$ while $x \in L_n$ are roots of pairwise disjoint subtrees $V_{\succeq x} = \{z \in \mathrm{V}(D_n) : x \preceq z\}$ of T. Working in $\widehat{\mathcal{D}}_n$ we apply horizontal collapsing successively to all pairs $u \neq w \in L_n$ with $\ell^{\mathrm{F}}_n(u) = \ell^{\mathrm{F}}_n(w)$. Let $\widehat{\mathcal{D}}_{n+1}$ be the resulting dag-like deduction of $\ell^{\mathrm{F}}_{n+1}(\mathrm{root}(\mathcal{D}_{n+1})) = \ell^{\mathrm{F}}_n(\mathrm{root}(\mathcal{D}_n)) = \ell^{\mathrm{F}}(\mathrm{root}(\mathcal{D}))$ with $\ell^{\mathrm{R}}_{n+1}(\mathrm{root}(\mathcal{D}_{n+1})) = \ell^{\mathrm{R}}_n(\mathrm{root}(\mathcal{D}_n)) = \ell^{\mathrm{R}}(\mathrm{root}(\mathcal{D}))$ and $\Gamma_{\widehat{\mathcal{D}}_{n+1}} \subseteq \Gamma_{\widehat{\mathcal{D}}_n} \subseteq \Gamma_{\mathcal{Q}}$ (see Conclusion 12).

Theorem 14 *For any encoded non-trivial tree-like* $\mathrm{NM}_{\to}$ *deduction $\mathcal{D} \in \mathbb{D}_{\to}$, $h(\mathcal{D}) > 0$, and compressed encoded dag-like* $\mathrm{NM}_{\to}$ *deduction $\mathcal{D}_{\mathrm{C}} \in \mathbb{D}_{\to}$ we have*

$$\boxed{|\mathcal{D}_{\mathrm{C}}| < h(\mathcal{D}) \cdot \phi(\mathcal{D})}$$

Moreover $\mathcal{D}$ and $\mathcal{D}_{\mathrm{C}}$ both have the same root formula $\alpha = \ell^{\mathrm{F}}(\mathrm{root}(\mathcal{D}))$, while all formulas and all open assumptions of $\mathcal{D}_{\mathrm{C}}$ occur as those of $\mathcal{D}$. In particular, if $\mathcal{D}$ is encoded tree-like $\mathrm{NM}_{\to}$ *proof of α, then $\mathcal{D}_{\mathrm{C}}$ is encoded dag-like proof of α whose size is polynomial in $|\alpha|$, provided that so are both $h(\mathcal{D})$ and $\phi(\mathcal{D})$.*

---

[8] $\langle x,y\rangle$ means "y is a son (or target) of the source x".

[9] Note that $\mathcal{D} = (\mathcal{D}_{\mathrm{U}})_{z\hookleftarrow y}$ holds true for encoded deductions exposed in Example 9.

Proof. Let $\mathcal{D} = \left\langle T, \ell^{\text{F}}, \ell^{\text{R}}, \ell^{\text{d}} \right\rangle$, $h(T) > 0$, and $\mathcal{D}_{\text{C}} = \left\langle D_{\text{C}}, \ell^{\text{F}}_{\text{C}}, \ell^{\text{R}}_{\text{C}}, \ell^{\text{d}}_{\text{C}} \right\rangle$. By Definition 13 we have $\text{v}(\mathcal{D}_{\text{C}}) = \text{v}(D_{\text{C}}) = \bigcup_{n=0}^{h(T)} F_n$, where $F_n = \left|\left\{\ell^{\text{F}}_n(x) : x \in L_n\right\}\right| \leq \phi(\mathcal{D})$, $F_0 < \phi(\mathcal{D})$. This yields $|\mathcal{D}_{\text{C}}| = |\text{v}(\mathcal{D}_{\text{C}})| < h(T) \cdot \phi(\mathcal{D}) = h(\mathcal{D}) \cdot \phi(\mathcal{D})$, as required. The rest follows immediately from Conclusion 12 and Definition 13. ∎

Corollary 15 *Let ∂ be any given tree-like* $\text{LM}_{\rightarrow}$ *deduction of α and $\mathcal{D} := F(\partial)$ the correlated encoded tree-like* $\text{NM}_{\rightarrow}$ *proof of α. Let $\mathcal{D}_{\text{C}}$ be the corresponding compressed encoded dag-like* $\text{NM}_{\rightarrow}$ *proof of α. Then $|\mathcal{D}_{\text{C}}| < 18(|\alpha|+1)^4$.*

Proof. By Lemma 5 we have

$$|\mathcal{D}_{\text{C}}| \leq h(F(\partial)) \cdot \phi(F(\partial)) < 18|\alpha|(|\alpha|+1)^2(|\alpha|+2) < 18(|\alpha|+1)^4.$$

∎

Corollary 16 ***NP = PSPACE****, and hence* ***NP = coNP = PSPACE = NPSPACE****.*

Proof. Recall that the validity problem for minimal propositional logic is PSPACE-complete, cf. [11], [12] and [10]. Now by standard arguments (cf. e.g. [1]) Corollary 15 shows that it is a NP problem. Indeed, consider any given purely implicational formula α. By Claim 2, α is valid in the minimal logic iff there exists a tree-like $\text{LM}_{\rightarrow}$ deduction ∂ of α. Hence, by Lemma 9 and Corollary 15, α is valid in the minimal logic iff we can "guess" an encoded dag-like proof $\mathcal{D}_{\text{C}}$ of α, in $\text{NM}_{\rightarrow}$, whose size $|\mathcal{D}_{\text{C}}|$ is polynomial in $|\alpha|$. Moreover, by Lemma 9, the assertion '$\mathcal{D}_{\text{C}}$ *is encoded dag-like* $\text{NM}_{\rightarrow}$ *proof of* α' is decidable in polynomial time with respect to $|\mathcal{D}_{\text{C}}|$, and hence also $|\alpha|$. Thus the existence of encoded dag-like $\text{NM}_{\rightarrow}$ proof of α is verifiable in polynomial time by a nondeterministic algorithm, and hence so is the problem of minimal validity of α, Q.E.D. (By Lemma 8, **NP = PSPACE** is also inferred by "guessing" a desired encoded version $\mathcal{D}_{\text{C}}$ of plain $|\alpha|$-polysize dag-like $\text{NM}_{\rightarrow}$ proof ∂_{C} that is obtained by a simpler, but less constructive, ℓ^{d}-free mapping $\partial \vdash \alpha \hookrightarrow \partial_{\text{C}} \vdash \alpha$.) ∎

Remark 17 *Our proof also shows that there is a polynomial time algorithm that for any input tree-like* $\text{LM}_{\rightarrow}$ *deduction ∂ of α outputs an encoded dag-like* $\text{NM}_{\rightarrow}$ *proof of α whose size is polynomial in $|\alpha|$.*

Let us dig deeper. According to [6], a suitable proof search procedure in $\text{LM}_{\rightarrow}$ (originally, LG) yields an $O(n \log n)$–SPACE decision algorithm for the validity in minimal (resp. intuitionistic) propositional logic. A PTIME algorithm is hardly possible, since $(\text{M}E \rightarrow\rightarrow)$ is the (only) non-invertible rule in $\text{LM}_{\rightarrow}$ (cf. [6] with regard to $(\text{G}E \rightarrow\rightarrow)$ in LG). More precisely, having arrived at presumably valid conclusion $\Gamma, (\alpha \rightarrow \beta) \rightarrow \gamma \Longrightarrow q$ of $(\text{M}E \rightarrow\rightarrow)$ of minimal priority (so it can't be inferred by other rules of $\text{LM}_{\rightarrow}$) we conclude that either $(\alpha \rightarrow \beta) \rightarrow \gamma$ or some $(\alpha' \rightarrow \beta') \rightarrow \gamma'$ occurring in Γ is the principal formula of (M$E \to\to$) in question – the only problem is how to find the appropriate one. This obstacle complicates a more efficient bottom-up search (even) for a regular LM$_{\to}$ proof (see Lemma 3 (5)), whose steps are uniquely determined up to the choice of the antecedent formula $(\alpha \to \beta) \to \gamma$ that, as just mentioned, is appropriate for the inverse validity of (M$E \to\to$). Summing up we arrive at

Criterion 18 ***P* = *NP***, *and hence* ***P* = *PSPACE*** *and/or* ***P* = *NPSPACE***, *holds iff the following problem is in* ***P***. *For any sequences of pairwise distinct variables* $p_1, \cdots, p_k, q$ *and formulas* $(\alpha_1 \to \beta_1) \to \gamma_1, \cdots, (\alpha_n \to \beta_n) \to \gamma_n$ *with* $(\forall i = 1, \cdots, n)\, q \in \mathrm{VAR}(\gamma_i)$, $n > 1$, *we let*

$$\varphi \;:= p_1 \to \cdots \to p_k \to ((\alpha_1 \to \beta_1) \to \gamma_1) \to \cdots \to ((\alpha_n \to \beta_n) \to \gamma_n)$$
$$\varphi^- \;:= p_1 \to \cdots \to p_k \to ((\alpha_1 \to \beta_1) \to \gamma_1) \to \cdots \to \left((\alpha_{n-1} \to \beta_{n-1}) \to \gamma_{n-1}\right)$$

where $\to$ *is right-associative.*

Problem: *Is* $\varphi^- \to \alpha_n \to (\beta_n \to \gamma_n) \to \beta_n$ *valid in minimal propositional logic, provided that so is* $\varphi \to q$ *?*

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4 Appendix A

A required loose upper bound $\text{ssf}(\xi) \leq (|\xi|+1)^2$ is proved by induction on $|\xi|$, as follows. Recall the recursive clauses 1–3:

1. $\text{ssf}(p) := 1$.
2. $\text{ssf}(p \to \alpha) := 2 + \text{ssf}(\alpha)$.
3. $\text{ssf}((\alpha \to \beta) \to \gamma) := 1 + \text{ssf}(\alpha \to \beta) + \text{ssf}(\beta \to \gamma) - \text{ssf}(\beta)$.

- Basis of induction. Suppose $|\xi| = 0$. Hence $\xi = p$ and $\text{ssf}(\xi) = 1 = (|\xi|+1)^2$, since $|p| = 0$.
- Induction step. Suppose $|\xi| > 0$. Hence $\xi = \alpha \to \beta$.
  - If $|\alpha| = 0$, then $\alpha = p$ and $\text{ssf}(\xi) = 2 + \text{ssf}(\beta) \underset{I.H.}{\leq} 2 + (|\beta|+1)^2$ $< (|\beta|+2)^2 = (|\xi|+1)^2$.
  - Otherwise $\alpha = \gamma \to \delta$ and $\xi = (\gamma \to \delta) \to \beta$. If $|\delta| = 0$, then $\delta = p$ and $\text{ssf}(\xi) = 1 + \text{ssf}(\alpha) + \text{ssf}(p \to \beta) - \text{ssf}(p) = 2 + \text{ssf}(\alpha) + \text{ssf}(\beta)$ $\underset{I.H.}{\leq} 2 + (|\alpha|+1)^2 + (|\beta|+1)^2 < (|\alpha|+|\beta|+1)^2 = (|\xi|+1)^2$.
  - Otherwise $\delta = \zeta \to \eta$ and $\xi = (\gamma \to (\zeta \to \eta)) \to \beta$. If $|\eta| = 0$, then $\eta = p$ and $\text{ssf}(\xi) = 1 + \text{ssf}(\alpha) + \text{ssf}((\zeta \to p) \to \beta) - \text{ssf}(\zeta \to p)$ $= 2 + \text{ssf}(\alpha) + \text{ssf}(p \to \beta) - \text{ssf}(p) = 3 + \text{ssf}(\alpha) + \text{ssf}(\beta)$ $\underset{I.H.}{\leq} 3 + (|\alpha|+1)^2 + (|\beta|+1)^2 < (|\alpha|+|\beta|+1)^2 = (|\xi|+1)^2$.
  - ⋯ ⋯ ⋯ ⋯ ⋯ ⋯ ⋯ ⋯ ⋯ ⋯ ⋯ ⋯
  - Eventually we arrive at $\alpha = \gamma_1 \to \cdots \to \gamma_n \to p$ (right-associative) and $\text{ssf}(\xi) = \text{ssf}(\alpha \to \beta) = n + 1 + \text{ssf}(\alpha) + \text{ssf}(\beta)$ $\underset{I.H.}{\leq} n + 1 + (|\alpha|+1)^2 + (|\beta|+1)^2 < (|\alpha|+|\beta|+1)^2 = (|\xi|+1)^2$.

This completes the proof of Lemma 3 (4).

5 Appendix B

Aside our trivial Example 6 let us consider a more sophisticated horizontal compression that deals with Haeusler's tautologies in minimal logic [5].

Definition 19 *For any $n > 0$ let $\varphi_n := \xi_n \to p$, where ξ_n is defined as follows by recursion on n, where $\to$ is left-associative (p, q_1, q_2, $\ldots$ being pairwise distinct propositional variables).*

1. $\xi_1 := q_1 \to p \to q_1 \to q_1 \to p$.
2. $\xi_{n+1} := q_{n+1} \to \xi_n \to q_{n+1} \to q_{n+1} \to \xi_n$.

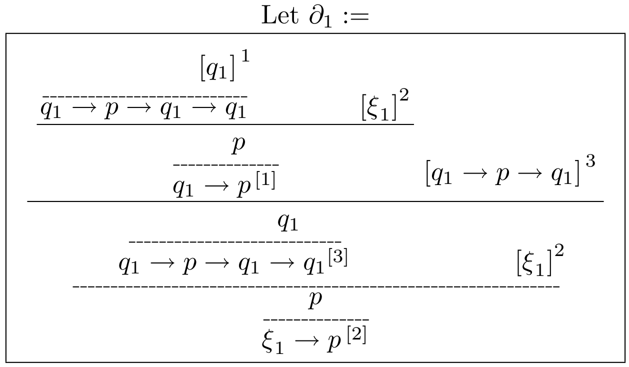
Theorem 20 *[5] Any normal tree-like proof of φ_n in $\mathrm{NM}_{\to}$ requires at least 2^n (discharged) occurrences of the assumption ξ_n.*

So the size of normal tree-like $\mathrm{NM}_{\to}$ proofs of φ_n is exponential in n. But

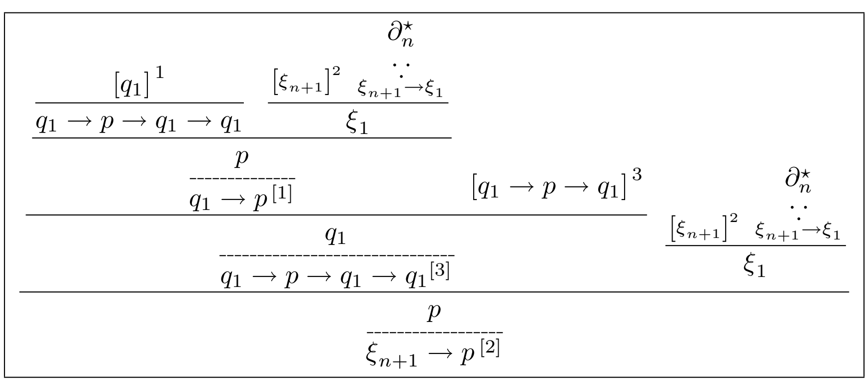
Theorem 21 *There are normal tree-like $\mathrm{NM}_{\to}$ proofs of φ_n whose horizontal dag-like compressions have size $< \min\left\{42n^2 + 13n + 1, 12\left(2^n - 1\right)\right\}$, thus being merely quadratic in n.*

Proof. For brevity we consider plain (i.e. unencoded) natural deductions. Following [5] we define normal tree-like proofs ∂_n of φ_n by recursion on n.

Let $\partial_1 :=$



and $\partial_{n+1} :=$



where $\partial_n^\star := \partial_n\left[p := \xi_1, \{q_i := q_{i+1}\}_{0<i\leq n}\right]$; note that according to [5] we have $\xi_{n+1} = \xi_n\left[p := \xi_1, \{q_i := q_{i+1}\}_{0<i\leq n}\right]$. Let us count the crucial parameters

$h(\partial_n)$, $\phi(\partial_n)$ and $|\partial_n|$ (= total number of all nodes occurring in ∂_n). Clearly $h(\partial_1) = 8$, $\phi(\partial_1) = 7$ and $|\partial_1| = 11$, while for each $n > 0$ we have $h(\partial_{n+1}) = h(\partial_1) + h(\partial_n) - 1$, $\phi(\partial_{n+1}) = \phi(\partial_1) + \phi(\partial_n) - 1$ and $|\partial_{n+1}| = |\partial_1| + 2(|\partial_n| + 1)$. This yields $h(\partial_n) = 7n+1$, $\phi(\partial_n) = 6n+1$ and $|\partial_n| = 12 \cdot 2^n - 13$. Now consider the corresponding dag-like horizontal compression $(\partial_n)_{\mathrm{C}}$. By Theorem 14 we have $|(\partial_n)_{\mathrm{C}}| < h(\partial_n) \cdot \phi(\partial_n) = 42n^2 + 13n + 1$, while $|(\partial_n)_{\mathrm{C}}| \le |\partial_n| < 12(2^n - 1)$ by the definition of dag-like compression. This yields a required estimate

$$|(\partial_n)_{\mathrm{C}}| < \min\left\{42n^2 + 13n + 1, 12(2^n - 1)\right\} = \begin{cases} 12(2^n - 1) & \text{if} \quad n \le 7, \\ 42n^2 + 13n + 1 & \text{if} \quad n > 7. \end{cases}$$

Thus for $n > 7$ the horizontal dag-like compression essentially reduces the size of ∂_n. Note that the correlated collapsing must be applied already for $n = 4$ at height-level 20. To recognize this consider the lists $H_n = \left[l_0^n, l_1^n, \cdots, l_{h(\partial_n)}^n\right]$ where $l_k^n := |\{x \in \partial_n : h(x) = k\}|$, $0 \le k \le h(\partial_n)$, $n > 0$. For $n = 4$ we have

$$H_4 = \left[\begin{array}{c} 1,1,2,3,3,3,5,6,4,7,9,7,10,14,13,12,20,20,18,18,\mathbf{28},19,24,18, \\ 24,12,21,12,11,5,10,5,2,1,2,1 \end{array} \right]$$

whose maximal (boldface) element is $l_{20}^4 = 28$. This means that ∂_4 contains exactly 28 nodes at the height-level 20. However, the total number of pairwise distinct formulas occurring in ∂_4 is merely $\phi(\partial_4) = 6 \cdot 4 + 1 = 25 < 28$. Hence $(\partial_4)_{\mathrm{C}}$ requires at least three collapsing steps to be made at the level 20. This should obviously reduce the size of ∂_4 and strengthen our general estimate for $(\partial_4)_{\mathrm{C}}$ (we omit the details). A similar argument applies to all $4 \le n < 8$. ∎

Remark 22 *Every $(\partial_n)_{\mathrm{C}}$ under consideration is further compressible to a normal dag-like proof of φ_n whose size in quadratic in n. For (by contrast to the well-known tree-like case) any dag-like natural deduction is normalizable without increasing the size. To this end it suffices to combine standard normalization procedure with the collapsing of equal assumptions. The collapsing in question is carried out as in Definition 11, except applying it as long as possible to distinct leaves $u \neq w$ supplied with equal discharged assumptions. Clearly this does not increase the size of deduction. Moreover in the resulting dag-like deduction every discharged assumption occurs only once. Hence every subsequent reduction includes only one substitution, which does not increase the size, either (cf. analogous argument used in [8]).*